Discrete Applied Mathematics 14 (1986) 1-9
North-Holland


# ON AN EXTREMAL PROBLEM CONCERNING THE INTERVAL NUMBER OF A GRAPH

Thomas ANDREAE
*Freie Universität Berlin, Institut für Mathematik II, Arnimallee 3, D-1000 Berlin 33, FRG*

Received 23 January 1985
Revised 7 June 1985

The interval number of a simple undirected graph $G$, denoted $i(G)$, is the least non-negative integer $r$ for which we can assign to each vertex in $G$ a collection of at most $r$ intervals on the real line such that two distinct vertices $v$ and $w$ of $G$ are adjacent if and only if some interval for $v$ has non-empty intersection with some interval for $w$. J.R. Griggs proved that $i(G) \leq k$ for each graph $G$ with fewer than $4k$ vertices ($k \in \mathbb{N}$). In addition, a result of W.T. Trotter and F. Harary shows that there exists a graph on $4k$ vertices which has interval number $k+1$: these authors determined $i(K_{n,m})$ for each complete bipartite graph $K_{n,m}$ and, in particular, they found that $i(K_{2k,2k}) = k+1$ ($k \in \mathbb{N}$). Here it is proved that $i(G) \leq k$ still holds for each graph $G$ on $4k$ vertices, except for $K_{2k,2k}$ ($k \in \mathbb{N}$). This settles a conjecture of Trotter.

## 1. Introduction

The interval number of a (finite, simple and undirected) graph $G$ was introduced to generalize interval graphs. A graph $G$ is called an *interval graph* if it is possible to assign to each vertex of $G$ an interval on the real line such that two distinct vertices of $G$ are adjacent if and only if the corresponding intervals have non-empty intersection. In [5,11], Trotter and Harary and, independently, Griggs and West generalized this concept by introducing the *interval number* $i(G)$ of a graph $G$ which is defined as the least non-negative integer $r$ for which we can assign to each vertex in $G$ a collection of at most $r$ intervals on the real line such that two distinct vertices $v$ and $w$ of $G$ are adjacent if and only if some interval for $v$ has non-empty intersection with some interval for $w$. Then interval graphs are exactly the graphs with interval number at most one. Note that in the above definitions it may be assumed w.l.o.g. that all intervals under consideration are closed and bounded.

Several authors have noticed that interval graphs and interval numbers are useful in scheduling and allocation problems. For the various applications of interval graphs, see for example the books of Golumbic [3] and Roberts [9]. Possible applications of interval numbers are shortly discussed in [6,8,10]. For example in [10], the authors point out that interval numbers may have applications concerning certain investigations on the structure of genes (see Chambon [2]). Recent results on interval numbers can be found in [7,8,10,12].

The concern of the present paper is to prove the following conjecture which is due

0166-218X/86/$3.50 

to Trotter (see [4]). A result of Griggs [4] states that $i(G) \leq k$ for each graph $G$ with fewer than $4k$ vertices ($k \in \mathbb{N}$). In a sense, this is sharp since $i(K_{2k,2k}) = k+1$ ($k \in \mathbb{N}$), which was shown by Trotter and Harary in [11]. Trotter's conjecture, and the result of the present paper, is that $i(G) \leq k$ still holds for each graph $G$ on $4k$ vertices, except for $K_{2k,2k}$ ($k \in \mathbb{N}$). Thus among the graphs with at most $4k$ vertices, $K_{2k,2k}$ is the unique graph $G$ for which $i(G)$ is maximal.

## 2. Terminology

We adopt the terminology of Griggs [4]; for basic graph-theoretical concepts and notations, see the book of Bondy and Murty [1]. All definitions and notations that are not explained in this paper can be found in one of these two references. Nevertheless, it is still necessary to provide some additional terminology. All graphs considered in this paper are finite, simple and undirected. The letter $G$ always denotes a graph. $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. A *representation of* $G$ is a mapping $f$ that assigns to each $v \in V(G)$ a finite set $f(v)$ of closed, bounded intervals on the real line $\mathbb{R}$ such that, for each pair of distinct $v, w \in V(G)$, $v$ and $w$ are adjacent in $G$ if and only if some interval of $f(v)$ has non-empty intersection with some interval of $f(w)$. For $F \subseteq E(G)$, $G[F]$ denotes the subgraph of $G$ that has $F$ as its edge set and, as its vertex set, all vertices of $G$ which are incident with some edge of $F$. If $f$ is a representation of $G[F]$, then $f$ will also (simply) be called a *representation of* $F$ ($F \subseteq E(G)$). In connection with representations $f$, we shall sometimes prefer to use more informal terminology which either is explained in [4] or is immediately clear from the context. For example, the expressions "$I$ is a $v$-interval" or "$I$ represents $v$" both mean the same, namely that $I \in f(v)$ for the particular representation $f$ which is under consideration.

Let $f$ be a representation of $G$ and let $v \in V(G)$. Further, let $b$ be a point on the real line. We say (terminology of [10]) that $v$ *has a broken end* $b$ *in* $f$ if there exists some $I \in f(v)$ such that (i) $b$ is an endpoint of $I$ and (ii) $b \notin J$ for each $J \in f(w)$ ($w \neq v$) and each $J \in f(v) \setminus \{I\}$. Let $I, J$ be closed, bounded intervals of the real line such that $I$ and $J$ have the same left endpoint and $I$ is properly contained in $J$. Then $J$ is called a *right extension* of $I$. Similarly, a *left extension* is defined. In the proof of our theorem, we shall carry out several times the following operation which changes a given representation $f^*$ into a new representation $f$. For some vertex $v$, we drop a certain $v$-interval $I$ from $f^*(v)$ and insert, instead of $I$, a new $v$-interval $J$ which is a right or left extension of $I$. (Doing this we will always take care that the resulting $f$ represents the same graph as $f^*$.) In this case, the described exchange will also be called a right or left extension of $I$, and, for simplicity, the new (extended) $v$-interval will again be denoted by $I$. As usual, $|M|$ denotes the cardinality of a set $M$. $\mathbb{N}$ denotes the set of positive integers.

## 3. Proof of Trotter's Conjecture

In this section, we assume that the reader is familiar with the details of Griggs [4]. The proof of the following Lemma 1 is omitted since the lemma can easily be proved by a construction which is similar to the constructions used by Trotter and Harary in their proof of [11, Theorem 2]. (For example, one can proceed as shown in Fig. 1 for $k=2$, where $a_j, b_j$ $(j=1,\dots,4)$ denote the vertices of $K_{4,4}$. Alternatively, a slight alteration of [6, Fig. 3] yields the same: all that is needed is to drop the rightmost interval from [6, Fig. 3].)

**Lemma 1.** *Let $x$ be an arbitrary vertex of $K_{2k,2k}$ $(k\in\mathbb{N})$. Then there exists a representation $f$ of $K_{2k,2k}$ such that $|f(v)|=k$ for each vertex $v\neq x$, $|f(x)|=k+1$ and, furthermore, $x$ has a broken end in $f$.*

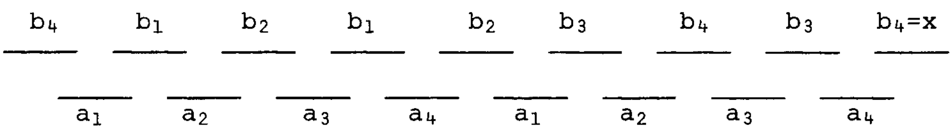


Fig. 1.

The proof of our theorem is based on the above Lemma 1 and on a refinement of the techniques introduced by Griggs in his proof of [4, Theorem 3]. Indeed, Griggs noticed in his paper [4, p. 38] that "the techniques used to prove Theorem 3 may shed light on this (Trotter's) conjecture". Furthermore, the following lemma is useful.

**Lemma 2.** *Let $G$ and $H$ be graphs which contain no triangles and assume that $V(G)=V(H)$ and $E(G)\subseteq E(H)$. Then $i(G)\leq i(H)$.*

We omit the proof since the argument presented in [4, Case 1] may serve as the proof of Lemma 2. Moreover, we mention that Lemma 2 can be used to shorten the first part of the proof of [4, Theorem 3]: by application of Lemma 2 one can easily show that the Cases 2 and 3 of the proof of [4, Theorem 3] need no separate consideration since it is sufficient to settle Case 4. (This will become apparent further down in the proof of our theorem.)

**Theorem.** *Let $|V(G)|=4k$ and $G\not\cong K_{2k,2k}$ $(k\in\mathbb{N})$. Then $i(G)\leq k$.*

**Proof.** The proof is by induction on $k$. For $k=1$, the theorem clearly holds. So assume that $k\geq 2$ and that the theorem is true for all values less than $k$. Let $n:=4k$. If $G$ is disconnected, then $i(G)\leq k$ since we can apply [4, Theorem 3] to each of the components of $G$. Thus let $G$ be connected. (Hereafter we no longer refer to the fact that $G$ is connected; however, the reader will notice that several times this fact is used implicitly.)

Our proof is organized similarly to the proof of Griggs [4, Theorem 3]. Starting from the top of page 39 of [4], we shall follow the line of the proof there. The reader will find that each part of the present proof has a counterpart in the proof of [4, Theorem 3]. There are even some parts which are, except for some minor alterations, the same as the corresponding parts in [4]. In this case, we leave the details of the proof to the reader. Recall that, throughout, familiarity with the details of [4] is assumed.

If $G$ is bipartite, then our theorem is well-known. (See [4, p. 38]. For completeness, the bipartite case is also settled below.) If $G$ contains an odd cycle, then we shall distinguish between certain subcases each of which depends on the following argument. Let $S$ be a subset of $V(G)$ such that $|S|=4$. Put $S':=V(G)\setminus S$ and let $G'$ be the graph that is induced by $S'$ in $G$. Let $F$ be the set of all edges in $G$ which are incident with at least one vertex of $S$; let further $F'$ be the set of edges of $G$ that have both endpoints in $V(G[F])$, and let $f$ be a representation of some $F''$ for which $F\subseteq F''\subseteq F'$. (Hereafter, we shall call a representation like this "a representation of all edges involving $S$"; the reader will find that in most parts of our forthcoming discussion $F=F''$.) Suppose that $f$ has the following three properties.

(p1) $|f(v)|\leq k$ for each $v\in S$.

(p2) $|f(v)|=1$ for each $v\in S'$ which has a neighbor in $S$.

(p3) There exists some vertex $x\in S'$ such that either $x$ is not in the domain of $f$ (i.e., $x$ has no neighbor in $S$) or $x$ has a broken end $b$ in $f$.

Then $i(G)\leq k$ can be obtained as follows. Note that $|V(G')|=4(k-1)$; thus, in case that $G'\not\cong K_{2(k-1),2(k-1)}$, we can use the induction hypothesis to find $i(G')\leq k-1$. Note further that (p1) and (p1) imply that $i(G)\leq \max(k, i(G')+1)$. Hence $i(G)\leq k$ if $G'\not\cong K_{2(k-1),2(k-1)}$. If, on the other hand, $G'\cong K_{2(k-1),2(k-1)}$, then pick a vertex $x\in S'$ as in (p3). By Lemma 1, there exists a representation $g$ of $G'$ such that $|g(x)|=k$, $|g(v)|=k-1$ $(v\neq x)$ and, in addition, $x$ has a broken end $c$ in $g$. If $x$ has no neighbor in $S$, then $i(G)\leq k$ clearly follows, since w.l.o.g. we may assume that no interval of $f$ intersects an interval of $g$. If $x$ has a broken end $b$ in $f$, then note that w.l.o.g. we can assume that $b$ is the rightmost point of $f$ and that $c$ is the leftmost point of $g$. In addition, we can choose $f$ and $g$ such that $c=b$. Now, in the obvious sense, the rightmost interval of $f$ and the leftmost interval of $g$ can be pasted together to obtain a single $x$-interval. This reduces by one the number of $x$-intervals involved in the present construction. Then $i(G)\leq k$ follows immediately.

Thus it remains to show how an appropriate $S\subseteq V(G)$ together with a corresponding representation $f$ can be found. In the following, we shall always use the letters $S, S', G', f, x$ in the sense in which they were introduced in the preceeding paragraphs. In particular, $f$ will always denote a representation of all edges involving $S$ such that $f$ has the desired properties (p1), (p2) and (p3). Moreover when we use the symbol $S'$, it will always be clear from the context to which particular set $S$ we refer.

**Case 1.** $G$ is bipartite. Edges can be added to $G$, if necessary, to make a complete bipartite graph $H$, say $H \cong K_{s,t}$ $(s+t=4k)$. Then $i(G) \leq i(H)$ by Lemma 2. Application of [11, Theorem 2] yields $i(H)=\lceil (st+1)/(s+t) \rceil$, which, in case that $s \neq t$, implies that $i(G) \leq k$. If $s=t=2k$, then (because $G \not\cong K_{2k,2k}$) there exists an edge $e$ of $H$ such that $G$ is a subgraph of $H-e$. By Lemma 2, $i(G) \leq i(H-e)$. This yields $i(G) \leq k$, since $i(H-e) \leq k$. (This can easily be shown by a method similar to the one which yields Lemma 1: for example, deleting the rightmost interval from Fig. 1 yields an appropriate representation of $K_{4,4}-e$.)

In the proof of [4, Theorem 3], the following two cases where considered.

**Case 2.** $G$ contains an odd cycle $C$ on $j \geq 9$ vertices and no smaller odd cycles.

**Case 3.** $G$ contains a 7-cycle $C$ and no smaller odd cycles.

In these cases let $j=|V(C)|$ and label the vertices of $C$ by $1,2,3,\ldots,j$ in order along the cycle. Then there is no edge of $G$ between the vertices 1 and 5, and, furthermore, there is no vertex in $G$ which is adjacent to both 1 and 5. (In either case, there would be an odd cycle of $G$ which has size less than $j$.) Let $H$ be the graph that results from $G$ by adding an edge between 1 and 5. Then $H$ contains a 5-cycle and no triangle. In particular, $i(G) \leq i(H)$ by Lemma 2. Hence the Cases 2 and 3 are settled in case that we have settled the following.

**Case 4.** $G$ contains a 5-cycle and no smaller odd cycles. We can proceed as in the corresponding case of [4] except for some modifications which are described below. We adopt all notations of [4, Case 4]; in particular, $S=\{1,2,3,4\}$ is defined as in [4, Case 4]. We also adopt the whole Fig. 4 of [4], i.e., the representation of the vertices of $S$ remains the same as that of [4, Case 4]. Further, the proof that in this construction at most $k$ intervals are used per vertex in $S$ can still be carried out as in [4]. (Checking this, the reader will find that the inequality $|P_4| \leq 2k-3$, as well as its counterpart $|P_1| \leq 2k-3$, is still valid if $n=4k$ instead of $n=4k-1$. Note further that the statement made in [4, p. 41] that $|P_4| \leq 2k-3$ holds "with equality only if $|P_5|=2k-3$" is no longer true for $n=4k$ instead of $n=4k-1$; but this does not affect the present proof since this statement is a casual remark of [4] which is nowhere used.)

To represent the edges between vertices of $S$ and $S'$ a modification of [4] is necessary. For $j=1,2,3,4$ let $I_j$ be the unique $j$-interval of [4, Fig. 4] which has non-empty intersection with some other interval of [4, Fig. 4]. Let $d$ be the left endpoint of the leftmost interval in [4, Fig. 4]. We construct the desired representation $f$ in two steps. First, represent all edges between vertices of $S$ and $S'$ as described in [4], but doing this, obey the following additional rule. For vertices $v$ outside $S$ adjacent to only one single $j \in S$, put the corresponding $v$-interval inside $I_j$. This yields a representation $f^*$ of all edges involving $S$. Note that $P_{13} \cup P_{14} \neq \emptyset$ since $5 \in P_{14}$. Among all $y$-intervals of $f^*$ for which $y \in P_{13} \cup P_{14}$, let $I$ be the one that is leftmost. Then, by an appropriate left extension of $I$, it can be achieved that $d$ is an inner point of $I$. Note that this yields a representation $f$ of all edges involving $S$ which has all desired properties; in particular, (p3) holds since $I$ has a broken end in $f$.

**Case 5.** $G$ contains a triangle. This case is split into three subcases bearing the same headlines as the cases (a), (b), (c) of [4, p. 42]. We shall use constructions similar to construction 1, 2 of [4]. In the discussion of these constructions as well as in all subcases of Case 5, we adopt the notations used in the corresponding parts of [4].

**Construction 1.** Adopting the suppositions of the first paragraph of [4, Construction 1], we shall distinguish between the two cases $P_{14} \cup P_{24} \neq \emptyset$ and $P_{14} \cup P_{24} = \emptyset$.

If $P_{14} \cup P_{24} \neq \emptyset$, then represent the edges between vertics in $S$ as shown in [4, Fig. 7]. Further, represent the edges between vertices of $S$ and vertices of $S'$ as described in [4], being careful that, for a certain $\upsilon \in P_{14} \cup P_{24}$, the correspond $\upsilon$-interval $I$ is placed across the rightmost gap between a 4-interval and a pair of 1- and 2-intervals. This yields a representation $f^*$ of all edges involving $S$. Now extend $I$ to the right so that it contains the right endpoint of the rightmost interval for 4. The right endpoint of $I$ is now a broken end as required by (p3). This yields $f$ as desired.

If $P_{14} \cup P_{24} = \emptyset$, then represent the edges between vertices in $S$ as shown in Fig. 2. Let $I_j$ be the left and $I_j'$ the right of the two $j$-intervals of Fig. 2 $(j = 1, 2, 3, 4)$.

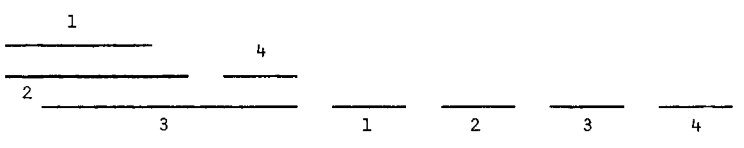


Fig. 2.

Recall that $P_{13} = P_{23} = P_{14} = P_{24} = \emptyset$. Consequently, there is no $\upsilon \in S'$ which has more than two neighbors in $S$, and each $\upsilon \in S'$ which has exactly two neighbors in $S$ must be either in $P_{12}$ or $P_{34}$. Pick a fixed $x \in S'$ which has at least one neighbor in $S$. If $x$ is adjacent to only a single $j \in S$, then represent the edge between $x$ and $j$ by placing an $x$-interval $I$ such that $I$ and $I_j'$ overlap and $I$ has empty intersection with all other intervals of Fig. 2. If on the other hand $x \in P_{\nu\mu}$ ($\nu = 1$ or 3 and $\mu = \nu + 1$), then represent the edges between $x$ and $S$ by placing an $x$-interval $I$ such that $I$ contains the common endpoint of $I_\nu$ and $I_\mu$ as an inner point; as above take care that $I$ has empty intersection with all intervals of Fig. 2 except for $I_\nu, I_\mu$ and, further, that no interval of Fig. 2 is contained in $I$.

Represent as follows all other vertices $\upsilon \in S'$ which have a neighbor in $S$. Depending on whether $\upsilon \in P_{12}$ or $\upsilon \in P_{34}$ or $\upsilon$ is adjacent to only a single $j \in S$, place a small $\upsilon$-interval inside the unused portion of $I_1 \setminus I_3$ or $I_4$ or $I_j'$.

Be careful to choose all $\upsilon$-intervals ($\upsilon \in S'$) involved in this construction to be pairwise disjoint. This yields $f$ as desired.

**Construction 2.** Adopting the suppositions of the first paragraph of [4, Construction 2], we distinguish between the two cases $|P_{14}| \le 4k-5$ and $|P_{14}| = 4k-4$.

If $|P_{14}| \le 4k-5$, then we can proceed as in [4, Construction 2]. This yields a representation $f^*$ of all edges involving $S$ such that (i) $f^*$ represents the vertices of $S$ either as shown in [4, Fig. 8] or as in a modified version of [4, Fig. 8] which results from [4, Fig. 8] by changing all 2-intervals to 3-intervals and vice versa, and (ii) $f^*$ represents the edges between $S$ and $S'$ as described in [4, Construction 2]. W.l.o.g. assume that the former of the two possibilities of (i) holds. It remains to change $f^*$ into a representation $f$ which has the desired properties.

Note that $|P_{14}| \le 4k-5$ implies $S' \setminus P_{14} \ne \emptyset$. If $N(x) = \emptyset$ for a certain $x \in S'$, then simply choose $f^*$ to be the desired $f$; this completes the construction as desired since (p3) trivially holds. If $N(x) = \{3\}$ for a certain $x \in S'$, then we may assume w.l.o.g. that the corresponding interval $I \in f^*(x)$ is placed inside the right side of the rightmost interval in [4, Fig. 8] and that $I$ is rightmost among all $v$-intervals $v \in S'$. Then an appropriate right extension of $I$ yields a representation $f$ with the desired properties. A similar argument applies to the case that $N(v) \ne \{3\}$ for each $v \in S'$ and $N(x) = \{2,3\}$ for a certain $x \in S'$. Thus we may assume that, for each $v \in S' \setminus P_{14}$, $N(v)$ is one of the sets $\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}$. Now, drop the rightmost 3-interval from $f^*$ and, in order to represent the edge between 2 and 3, insert a new 3-interval $J$ inside the leftmost 2-interval of $f^*$, taking care that $J$ intersects no other intervals beside this 2-interval and its corresponding 1-interval. Because $N(v) \ne \{3\}, \{2,3\}$ for each $v \in S'$, this results into a representation $f^{**}$ of all edges involving $S$. Using $f^{**}$ instead of $f^*$ and arguing as above, one can settle the following four cases.

(1) $N(x) = \{2\}$ for a certain $x \in S'$.
(2) $N(v) \ne \{2\}$ for each $v \in S'$ and $N(x) = \{2,4\}$ for a certain $x \in S'$.
(3) $N(x) = \{1\}$ for a certain $x \in S'$.
(4) $N(v) \ne \{1\}$ for each $v \in S'$ and $N(x) = \{1,3\}$ for a certain $x \in S'$.

Thus it remains to consider the case that, for each $v \in S' \setminus P_{14}$, $N(v)$ is one of the sets $\{4\}, \{1,2\}, \{3,4\}$. In this case, modify $f^{**}$ as follows. Change all 1-intervals of $f^{**}$ into 4-intervals and all 4-intervals into 1-intervals, and leave all 2-, 3- and $v$-intervals ($v \in P_{14}$) as in $f^{**}$. Thereafter, drop all $v$-intervals of $f^{**}$ for which $v \in S' \setminus P_{14}$ and insert new $v$-intervals, $v \in S' \setminus P_{14}$, as follows. If $N(v) = \{4\}$, then place a $v$-interval inside the left side of the leftmost 4-interval; if $N(v) = \{3,4\}(\{1,2\})$, then place $v$ inside the overlap of the two leftmost (rightmost) intervals. Then, by means of appropriate extensions, the remaining cases can be settled as above.

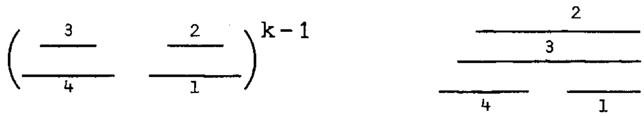


Fig. 3.

Next, assume that $|P_{14}|=4k-4$. In this case represent the edges between vertices in $S$ as shown in Fig. 3. Recall that $|P_{14}|=4k-4$ implies that, for each $v\in S'$, $N(v)$ is one of the sets $\{1,4\}, \{1,2,4\}, \{1,3,4\}$. Further, note that $t=2k-2$, which is exactly the number of 1–4 gaps in Fig. 3. Represent the edges between vertices of $S$ and $S'$ by placing intervals for both $v_i$ and $v_{i+t}$ over the same 1–4 gap ($i=1,\dots,t$) doing everything just as described in [4, Construction 2]. Note that, proceeding like this, we have no problems similar to the problem of [4, Construction 2] concerning the rightmost 1–4 gap of [4, Fig. 8]. This yields a representation $f^*$ of all edges involving $S$ such that $|f^*(v)|\le k(v\in S)$ and $|f^*(v)|=1(v\in S')$. It remains to properly modify $f^*$ into a representation $f$ for which (p3) holds. Let $J$ be the leftmost 3-interval of Fig. 3 and let $I$ be one of the two $x$-intervals ($x\in S'$) that are placed over the leftmost 1–4 gap of Fig. 3. If possible, choose $I$ such that $I$ and $J$ have non-empty intersection. Then, in case that $I\cap J\neq\emptyset$, an appropriate left extension of $I$ yields $f$ as desired. In case that $I\cap J=\emptyset$, just delete $J$ from Fig. 3 and then extend $I$. This also yields $f$ as desired. This completes the discussion of Construction 2.

**Case 5(a).** $G$ contains a triangle, but no kites or $K_4$'s. Note that the arguments used in the lines 19–31 [4, p. 45] remain valid for $n=4k$ instead of $n=4k-1$, except for some minor alterations as changing the equality $n-5=4k-6$ into $n-5=4k-5$. In particular, we can pick a vertex $6\in(P_{14}\cup P_{24})\cap(P_{25}\cup P_{35})$.

Since 6 cannot be adjacent to two or more of the vertices in $\{1,2,3\}$ (this would make a kite or a $K_4$), we conclude from $6\in(P_{14}\cup P_{24})\cap(P_{25}\cup P_{35})$ that 6 is adjacent to each of the vertices $2,4,5$ (Fig. 4). Consider the remaining $4k-6$ unlabelled vertices. Each of these vertices is adjacent to at most one of the vertices $\{1,2,3\}$. Consequently, there exists a vertex $j\in\{1,2,3\}$ such that $j$ is adjacent to at most $t:=\lfloor(4k-6)/3\rfloor$ of the unlabelled vertices. It follows that at most $t+2$ vertices outside $S=\{j,4,5,6\}$ are adjacent to $j$. Since $t+2\le 2k-2$, Construction 1 can be applied to this choice of $S$.

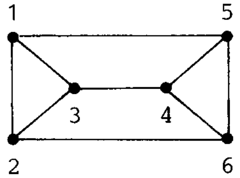

Fig. 4.

**Case 5(b).** $G$ contains a kite but no $K_4$. This case can be settled arguing as in the corresponding case of [4]; all that is needed is to make the following two minor changes which do not affect the validity of the proof. (i) In line 9 [4, p. 46], change $2k-3$ into $2k-2$ and (ii) in line 15 [4, p. 46], change $4k-6$ into $4k-5$.

**Case 5(c).** $G$ contains a $K_4$. For each $v\notin S=\{1,2,3,4\}$, let $N(v)$ be the set of neighbors of $v$ in $S$. Let $r:=\min\{|N(v)|: v\in S', N(v)\neq\emptyset\}$. W.l.o.g. we may assume the following, which can be achieved by a relabelling of the vertices in $S$ (if necessary).

(∗) There exists a vertex $x \in S'$ such that $N(x) = \{i \in S : 1 \leq i \leq r\}$ if $r \neq 3$ and $N(x) = \{2, 3, 4\}$ if $r = 3$.

Now, represent the edges between vertices of $S$ as shown in [4, Fig. 12] and represent all edges between $S$ and $S'$ as described in [4, Case 5(c)]. This yields a representation $f^*$ of all edges involving $S$. Pick a vertex $x$ as in (∗) such that the corresponding interval $I \in f^*(x)$ is leftmost among all $y$-intervals ($y \in S'$) for which $N(y) = N(x)$. Then it follows from (∗) together with the construction of $f^*$ (see [4, Fig. 12]) that $I$ is leftmost among all $v$-intervals, $v \in S'$. Further, if $r \neq 3$, then an appropriate left extension of $I$ yields $f$ as desired. In case that $r = 3$, delete the leftmost 1-interval of [4, Fig. 12] and then extend $I$. This also yields $f$ as desired. This completes the proof of the theorem. □

## References


[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (MacMillan Press, London, 1976).

[2] P. Chambon, Split genes, Sci. Amer. 244 (1981) 60–71.

[3] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).

[4] J.R. Griggs, Extremal values of the interval number of a graph, II, Discrete Math. 28 (1979) 37–47.

[5] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, SIAM J. Algebraic Discrete Methods 1 (1980) 1–7.

[6] L.B. Hopkins and W.T. Trotter Jr., A bound on the interval number of a complete multipartite graph, in: G. Chartrand, Y. Alavi, D.L. Goldsmith, L. Lesniak-Foster and D.R. Lick, eds., The Theory and Applications of Graphs (Wiley, New York, 1981) 391–407.

[7] L.B. Hopkins, W.T. Trotter jr. and D.B. West, The interval number of a complete multipartite graph, Discrete Appl. Math. 8 (1984) 163–187.

[8] C. Maas, Determining the interval number of a triangle-free graph, Computing 31 (1983) 347–354.

[9] F.S. Roberts, Discrete Mathematical Models (Prentice-Hall, Englewood Cliffs, 1976).

[10] E.R. Scheinerman and D.B. West, The interval number of a planar graph: Three intervals suffice, J. Combin. Theory (Series B) 35 (1983) 224–239.

[11] W.T. Trotter Jr. and F. Harary, On double and multiple interval graphs, J. Graph Theory 3 (1979) 205–211.

[12] D.B. West and D.B. Shmoys, Recognizing graphs with fixed interval number is NP-complete, Discrete Appl. Math. 8 (1984) 295–305.